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Synchronization of coupled memristive competitive BAM neural networks with different time scales

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ABSTRACT

In this paper, synchronization of coupled memristive competitive bidirectional associative memory (BAM) neural networks with different time scales is discussed. Two kinds of feedback controllers are designed such that the response system and the drive system can reach synchronization. By using the differential inclusions theory, and constructing a proper Lyapunov–Krasovskii functional, novel sufficient conditions are obtained to achieve asymptotical synchronization of competitive BAM neural networks. The proposed synchronization can be easily realized. An illustrative example is given to show the feasibility of our theoretical results.

1. Introduction

As a challenge and interesting area, brain science has attracted many theoretic researchers as well as experimental scientists all the time [1–5]. Many neuroscientists have tried to explore and uncover neural mechanisms of the brain, which have a strong potential to treat degenerative diseases and protect our brain in future. In addition, it can provide ideas for artificial intelligence and cognitive science. At the same time, some mathematicians and physists also took part in this field and brought in several significant works. As the basic unit of the human brain, brilliant neurons are connected by electrical synapses and chemical synapses so that they can complete information transmissions and cognitive functions. In the process, they constitute a large complex dynamical network, which results in abundant dynamical phenomena. At theoretical levels, various kinds of neuron models or neural network models have been proposed to simulate real neurons and electrophysiological characteristics, such as Hodgkin–Huxley neuron [1], McClloch–Pitts model [2], Hopfield neural networks [3], BAM neural networks [4], and Cohen–Grossberg neural networks [5]. From the viewpoint of dynamical systems, a neural network can be a nonlinear dynamical system which is composed of many simple neuronal elements, its structure and connection strengths have rather important influence on dynamical evolutions.

Because of potential applications of neural networks, dynamical behaviors of neural networks attracted a lot of theoretical researchers [6–11]. These earlier neural networks are generally considered to construct a network by fixed synaptic strengths. Hebb proposed that the strength of synapses in human brain neurons is changed (Hebb's rule) [12], which has promoted the development of theories and applications of neural network. So, considering dynamic synaptic weights is more plausible for information transmissions of biological networks. Dynamic neural networks with feedforward and feedback connections between neural layers have potential application in visual processing, pattern recognition, etc. In order to reflect the competitive and cooperative characteristic of neurons, Lemmon et al. [13] considered a class of laterally inhibited network modified by external input. This class of neural networks describes the dynamical behaviors of both activity levels, the short-term memory (STM) and the long term memory (LTM). Meyer-Bäse et al. [14,15] extended previous models and proposed a class of competitive neural networks with different time scales. Based on different time scales, the short-term memory (STM) is faster than the long term memory (LTM) ($0 < \varepsilon < 1$). So the short-term memory (STM) refers to the fast neural activities,

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and the long term memory (LTM) to the slow activities of unsupervised synaptic modifications by input. The general neural network equations describing the temporal evolution of the STM and LTM states for the ith neuron of an n neurons network are as follows:

$$STM: \ \varepsilon\dot{x}_i(t) = -x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + H_i s_i(t),$$
$$LTM: \ \dot{s}_i(t) = s_i(t) + f_i(x_i(t)), \ i = 1, 2, \ldots, n.$$

Related papers on synchronization dynamics of competitive neural networks [16–18].

As collective behaviors, consensus and synchronization are popularly in nature, society and neural networks [19–23]. Neuronal collective activities are considered to complete certain tasks [24] and other diversity of cognitive functions, such as pattern recognition [20]. Various studies show that brain pathologies and disorders are related to synchronization behaviors of neurons [25,26]. Especially, synchronization in neuronal systems or neural networks are very interesting and very important [20,24–27]. Researchers found that the memristor has good memory characteristic like brain [28]. Therefore, a lot of researchers considered the memristor instead of artifical synapse for simulating some function of the brain [29]. Using the advantage of a memristor, some researchers proposed memristive neural networks and their synchronized dynamical behavior was discussed [18,20–33]. Shi et al. [18] investigated synchronization problems of memristive neural networks with different time scales.

However, most works on synchronization of competitive neural networks consider external input and auto-feedbck connections [14–18], which can realize auto-associative memories. In 1988, Kosko [4] proposed bidirectional associative memory(BAM) neural networks which can be realize a hetero–associative memories. It can be successfully applied to image processing, pattern recognition and compressing data and it has attracted great interest of many researchers [4,6,11,32,33]. This is certain physiological significance and inspiration. For an example, the visual cortex is generally considered to consist of six layers, from low-level to high-level, each layer completes specific information processing tasks, and finally completes the whole information transmission. This model can be regarded as a simplified model for achieving basic function. As far as we know, there are a few of results considering competitive bidirectional associative memory(BAM) neural networks with different time scales. Based on the above discussions and inspired by bidirectional associated neural networks [4,6] and laterally inhibited neural networks [13–15], in this paper, we propose a double-layer memristor-based competitive bidirectional associative memory(BAM) neural network with different time scales as follows:

$$X-layer: \begin{cases} STM: \ \varepsilon\dot{x}_i(t) = -x_i(t) + \sum_{j=1}^{m} a_{ji}(x_i(t)) f_j(y_j(t)) \\ \qquad + \sum_{j=1}^{m} b_{ji}(x_i(t)) f_j(y_j(t-\tau(t))) + I_i s_i(t), \\ LTM: \ \dot{s}_i(t) = -s_i(t) + g_i(x_i(t)), \ i = 1, 2, \ldots, n, \end{cases} \tag{1a}$$

$$Y-layer: \begin{cases} STM: \ \varepsilon\dot{y}_j(t) = -y_j(t) + \sum_{i=1}^{n} c_{ij}(y_j(t)) g_i(x_i(t)) \\ \qquad + \sum_{i=1}^{n} d_{ij}(y_j(t)) g_i(x_i(t-\tau(t))) + J_j r_j(t), \\ LTM: \ \dot{r}_j(t) = -r_j(t) + f_j(y_j(t)), \ j = 1, 2, \ldots, m, \end{cases} \tag{1b}$$

where $0 < \varepsilon < 1$,n denotes the number of neurons of the X-layer, and m denotes the number of neurons of the Y–layer. $x(t) = (x_1(t), \ldots, x_n(t))^T, x_i(t)$ is the neurons' fast neural activities of the X–layer, $y(t) = (y_1(t), \ldots, y_m(t))^T$ describes the neurons' fast neural activities of the Y–layer. $f_j(y_j(t))$ and $g_i(x_i(t))$ are the output of neurons of the X–layer and the Y–layer, respectively; furthermore, $f(y(t)) = (f_1(y_1(t)), \ldots, f_m(y_m(t)))^T$ and $g(x(t)) = (g_1(x_1(t)), \ldots, g_n(x_n(t)))^T$. $s_i(t)$ and $r_j(t)$ are correspondingly the slow activity of unsupervised synaptic efficiencies, respectively; furthermore, $s(t) = (s_1(t), \ldots, s_n(t))^T$ and $r(t) = (r_1(t), \ldots, r_m(t))^T$. I_i is the strength of the external stimulus for the X-layer and J_j is the strength according to the Y-layer. $a_{ji}, \ c_{ij}$ represent the connection weight between the ith neuron and the jth neuron, respectively; $b_{ji}, \ d_{ij}$ denote the synaptic weight of the delayed feedback, describing the dynamical efficiency of the synaptic strength between the X-layer and the Y-layer, respectively. Based on the memristor's feature, the current–voltage relationship can be generally described as follows:

$$a_{ji}(x_i(t)) = \begin{cases} \hat{a}_{ji}, & |x_i(t)| > T_i, \\ \check{a}_{ji}, & |x_i(t)| \leqslant T_i, \end{cases} \qquad b_{ji}(x_i(t)) = \begin{cases} \hat{b}_{ji}, & |x_i(t)| > T_i, \\ \check{b}_{ji}, & |x_i(t)| \leqslant T_i, \end{cases}$$

$$c_{ij}(y_j(t)) = \begin{cases} \hat{c}_{ij}, & |y_j(t)| > T_j, \\ \check{c}_{ij}, & |y_j(t)| \leqslant T_j, \end{cases} \qquad d_{ij}(y_j(t)) = \begin{cases} \hat{d}_{ij}, & |y_j(t)| > T_j, \\ \check{d}_{ij}, & |y_j(t)| \leqslant T_j, \end{cases}$$

in which the switching jumps $T_i, \ T_j > 0, \hat{a}_{ji}, \check{a}_{ji}, \hat{b}_{ji}, \check{b}_{ji}, \hat{c}_{ij}, \check{c}_{ij}, \hat{d}_{ij}, \check{d}_{ij}$ are all constant numbers and $\tau(t)$ corresponds to the transmission time-varying delay.

Remark 1. In Eqs. (1a) and (1b), we know $0 < \varepsilon < 1, x_i(t)$ (or $y_j(t)$) is faster than $s_i(t)$(or $r_j(t)$) from this time-scale. So $x_i(t)$ (or $y_j(t)$) refers to fast neural activities (STM) and $s_i(t)$ (or $r_j(t)$) to slow neural activities(LTM).

Remark 2. The competitive memristive neural network model (1) is basically a state-dependent nonlinear switching dynamical system. It extends traditional neural network models, such as BAM neural networks [6,11,26].

The main contributions of this paper are as follows:

(i) we extend competitive neural networks with different time scales to bidirectional associative memory neural networks with different time scales (hetero-associative memory neural network) and mainly proposed a double-layer memristor-based competitive bidirectional associative memory (BAM) neural network with different time scales.
(ii) we give two kinds of control design to realize synchronization of competitive memristive BAM neural networks with different scales. Based on the analysis of our results, it is theoretically easier to satisfy synchronization for a controller with delay-independence and it can be easily applied in practice.

The rest of this paper is organized as follows. In Section 2, we introduce some notations, definitions and some preliminary results. In Section 3, we present sufficient conditions for synchronizaiton of system (1) under two kinds of controller. Finally, in Section 4, an example illustrates the feasibility of our results.

2. Preliminaries

In the following, we give the notations used in this paper. The solutions of all systems are considered in Filippov's sense [34]. R^n and $R^{n\times n}$ denote the n-dimensional Euclidean space and the set of all $n\times n$ real matrices, respectively. $P>0$ means a real positive definite matrix. In the Banach space of all continuous functions $C([-\tau,0],R^n)$ equipped with the norm defined by $\|\phi\| = sup_{-\tau\leqslant t\leqslant 0}\left[\sum_{i=1}^{n}|\phi_i(t)|^2\right]^{1/2}$ for all $\phi=\left(\phi_{(t),\dots,\phi_n(t)}\right)\in C([-\tau,0],R^n)$, $co[\underline{a},\overline{a}]$ denotes the convex hull. For the vector $x(t)=(x_1(t),\dots,x_n(t))^T\in R^n$, $\|x\|$ denotes the Euclidean vector norm, $\|x\|=\left[\sum_{i=1}^{n}|x_i(t)|^2\right]^{1/2}$.

To obtain solutions of system (1), we furthermore assume that.(H1) There exists a diagonal matrix $L=diag(l_1,\dots,l_n),\eta=diag(\eta_1,\dots,\eta_m)$ satisfying

$$0\leqslant\frac{|g_i(x)-g_i(y)|}{|x-y|}\leqslant l_i,$$

$$0\leqslant\frac{|f_j(x)-f_j(y)|}{|x-y|}\leqslant \eta_j,$$

for any $x(t),y(t)\in R,\ i=1,\dots,n;\ j=1,\dots,m$.(H2) There exist positive constants τ,γ such that $0<\tau(t)\leqslant\tau,\dot{\tau}(t)\leqslant\gamma<1$.

In what follows, we introduce lemma and some definitions below [18,34]:

Lemma 1. *For any vector x, $y\in R^n$ and a positive constant a, the following matrix inequality holds:*

$$2x^Ty\leqslant\ ax^Tx+a^{-1}y^Ty$$

Definition 1. $E\subset R^n, x\mapsto F(x)$ is called a set-valued map from $E\hookrightarrow R^n$, if for all $x\in E$, there is a corresponding nonempty set $F(x)\subset R^n$.

Definition 2. For the system $\frac{dx}{dt}=g(x),x\in R^n$, with discontinous right-hand sides, a set-valued map $\phi(x)$ is defined as

$$\phi(x)=\bigcap_{\delta>0}\bigcap_{\mu(N)=0}\overline{co}[g(B(x,\delta))\setminus N]$$

where $\overline{co}[E]$ is the closure of the convex hull of the set $E, B(x,\delta)=\{y:\|y-x\|\leqslant\delta\}$ and $\mu(N)$ is a Lebesgue measure of the set N. A solution in Filippov's sense of the Cauchy problem for this system with the initial condition $x(0)=x_0$ is an absolutely continuous function $x(t),t\in[0,T]$, which satisfies $x(0)=x_0$ and the differential inclusion [35]:

$$\frac{dx}{dt}\in\phi(x)\quad for\ \ a.e.\ \ t\in[0,T].$$

Based on the theories of set-valued maps and differential inclusions above, the memristor-based neural network (1) can be rewritten as the following differential inclusion:

$$X-layer:\begin{cases}STM:\ \varepsilon\dot{x}_i(t)\in-x_i(t)+\sum_{j=1}^{m}co\left[\underline{a}_{ji},\overline{a}_{ji}\right]f_j(y_j(t))\\ \qquad+\sum_{j=1}^{m}co\left[\underline{b}_{ji},\overline{b}_{ji}\right]f_j(y_j(t-\tau(t)))+I_is_i(t),\\ LTM:\ \dot{s}_i(t)=-s_i(t)+g_i(x_i(t)),\ i=1,2,\dots,n,\end{cases}\tag{2A}$$

$$Y-layer:\begin{cases}STM:\ \varepsilon\dot{y}_j(t)\in-y_j(t)+\sum_{i=1}^{n}co\left[\underline{c}_{ij},\overline{c}_{ij}\right]g_i(x_i(t))\\ \qquad+\sum_{i=1}^{n}co\left[\underline{d}_{ij},\overline{d}_{ij}\right]g_i(x_i(t-\tau(t)))+J_jr_j(t),\\ LTM:\ \dot{r}_j(t)=-r_j(t)+f_j(y_j(t)),\ j=1,2,\dots,m,\end{cases}\tag{2B}$$

where $\overline{a}_{ji}=max\{\hat{a}_{ji},\check{a}_{ji}\},\ \underline{a}_{ji}=min\{\hat{a}_{ji},\check{a}_{ji}\},\ \overline{b}_{ji}=max\left\{\hat{b}_{ji},\check{b}_{ji}\right\},\ \underline{b}_{ji}=min\left\{\hat{b}_{ji},\check{b}_{ji}\right\},\overline{c}_{ij}=max\{\hat{c}_{ij},\check{c}_{ij}\},\underline{c}_{ij}=min\{\hat{c}_{ij},\check{c}_{ij}\},\overline{d}_{ij}=max\left\{\hat{d}_{ij},\check{d}_{ij}\right\},\ \underline{d}_{ij}=min\left\{\hat{d}_{ij},\check{d}_{ij}\right\}$.

From [9–11,34–36], there exist $\tilde{a}_{ji}\in co[\hat{a}_{ji},\check{a}_{ji}],\tilde{b}_{ji}\in co\left[\hat{b}_{ji},\check{b}_{ji}\right]$, $\tilde{c}_{ij}\in co[\hat{c}_{ij},\check{c}_{ij}]$, and $\tilde{d}_{ij}\in co\left[\hat{d}_{ij},\check{d}_{ij}\right]$ such that

$$X-layer:\begin{cases}STM:\ \varepsilon\dot{x}_i(t)=-x_i(t)+\sum_{j=1}^{m}\tilde{a}_{ji}f_j(y_j(t))\\ \qquad+\sum_{j=1}^{m}\tilde{b}_{ji}f_j(y_j(t-\tau(t)))+I_is_i(t),\\ LTM:\ \dot{s}_i(t)=-s_i(t)+g_i(x_i(t)),\ i=1,2,\dots,n,\end{cases}\tag{3a}$$

$$Y-layer:\begin{cases}STM:\ \varepsilon\dot{y}_j(t)=-y_j(t)+\sum_{i=1}^{n}\tilde{c}_{ij}g_i(x_i(t))\\ \qquad+\sum_{i=1}^{n}\tilde{d}_{ij}g_i(x_i(t-\tau(t)))+J_jr_j(t),\\ LTM:\ \dot{r}_j(t)=-r_j(t)+f_j(y_j(t)),\ j=1,2,\dots,m.\end{cases}\tag{3b}$$

This paper considers system (2) or (3) as the drive system and the corresponding response system is as follows:

$$X-layer:\begin{cases}STM:\ \varepsilon\dot{\tilde{x}}_i(t)\in-\tilde{x}_i(t)+\sum_{j=1}^{m}co\left[\underline{a}_{ji},\overline{a}_{ji}\right]f_j(\tilde{y}_j(t))\\ \qquad+\sum_{j=1}^{m}co\left[\underline{b}_{ji},\overline{b}_{ji}\right]f_j(\tilde{y}_j(t-\tau(t)))+I_i\tilde{s}_i(t)+u_i(t),\\ LTM:\ \dot{\tilde{s}}_i(t)=-\tilde{s}_i(t)+g_i(\tilde{x}_i(t)),\ i=1,2,\dots,n,\end{cases}\tag{4a}$$

$$Y-layer:\begin{cases}STM:\ \varepsilon\dot{\tilde{y}}_j(t)\in-\tilde{y}_j(t)+\sum_{i=1}^{n}co\left[\underline{c}_{ij},\overline{c}_{ij}\right]g_i(\tilde{x}_i(t))\\ \qquad+\sum_{i=1}^{n}co\left[\underline{d}_{ij},\overline{d}_{ij}\right]g_i(\tilde{x}_i(t-\tau(t)))+J_j\tilde{r}_j(t)+\nu_j(t),\\ LTM:\ \dot{\tilde{r}}_j(t)=-\tilde{r}_j(t)+f_j(\tilde{y}_j(t)),\ j=1,2,\dots,m,\end{cases}\tag{4b}$$

There equivalently exist $\tilde{a}_{ji}\in co[\hat{a}_{ji},\check{a}_{ji}],\ \tilde{b}_{ji}\in co\left[\hat{b}_{ji},\check{b}_{ji}\right],\ \tilde{c}_{ij}\in co[\hat{c}_{ij},\check{c}_{ij}]$, and $\tilde{d}_{ij}\in co\left[\hat{d}_{ij},\check{d}_{ij}\right]$ such that[35,36],

$$X-layer:\begin{cases}STM:\ \varepsilon\dot{\tilde{x}}_i(t)=-\tilde{x}_i(t)+\sum_{j=1}^{m}\tilde{a}_{ji}f_j(\tilde{y}_j(t))\\ \qquad+\sum_{j=1}^{m}\tilde{b}_{ji}f_j(\tilde{y}_j(t-\tau(t)))+I_i\tilde{s}_i(t)+u_i(t),\\ LTM:\ \dot{\tilde{s}}_i(t)=-\tilde{s}_i(t)+g_i(\tilde{x}_i(t)),\ i=1,2,\dots,n,\end{cases}\tag{5a}$$

$$Y-layer:\begin{cases} STM:\ \varepsilon\dot{\tilde{y}}_j(t) = -\tilde{y}_j(t) + \sum_{i=1}^{n}\tilde{c}_{ij}g_i(\tilde{x}_i(t)) \\ \qquad + \sum_{i=1}^{n}\tilde{d}_{ij}g_i(\tilde{x}_i(t-\tau(t))) + J_j\tilde{r}_j(t) + \upsilon_j(t), \\ LTM:\ \dot{\tilde{r}}_j(t) = -\tilde{r}_j(t) + f_j(\tilde{y}_j(t)),\ j = 1,2,\ldots,m, \end{cases} \tag{5b}$$

where $y(t) \in R^n$ is the state vector of the response system, $u_i(t),\ \upsilon_j(t)$ is the controller to be designed for synchronization. Let the error $e_i^x(t) = \tilde{x}_i(t) - x_i(t), e_j^y(t) = \tilde{y}_j(t) - y_j(t), h_i^s(t) = \tilde{s}_i(t) - s_i(t)$ and $h_j^r(t) = \tilde{r}_j(t) - r_j(t)$. Then the error system is given as follows:

$$X-layer:\begin{cases} STM:\ \varepsilon\dot{e}_i^x(t) \in -e_i^x(t) + \sum_{j=1}^{m}co[\underline{a}_{ji},\overline{a}_{ji}]f_j\left(e_j^y(t)\right) \\ \qquad + \sum_{j=1}^{m}co\left[\underline{b}_{ji},\overline{b}_{ji}\right]f_j\left(e_j^y(t-\tau(t))\right) + I_ih_i^s(t) + u_i(t), \\ LTM:\ \dot{h}_i^s(t) = -h_i^s(t) + g_i(e_i^x(t)),\ i = 1,2,\ldots,n, \end{cases} \tag{6a}$$

$$Y-layer:\begin{cases} STM:\ \varepsilon\dot{e}_j^y(t) \in -e_j^y(t) + \sum_{i=1}^{n}co[\underline{c}_{ij},\overline{c}_{ij}]g_i(\tilde{x}_i(t)) \\ \qquad + \sum_{i=1}^{n}co\left[\underline{d}_{ij},\overline{d}_{ij}\right]g_i(e_i^x(t-\tau(t))) + J_jh_j^r(t) + \upsilon_j(t), \\ LTM:\ \dot{h}_j^r(t) = -h_j^r(t) + f_j\left(e_j^y(t)\right),\ j = 1,2,\ldots,m, \end{cases} \tag{6b}$$

Similarly, there equivalently exist $\tilde{a}_{ji} \in co[\hat{a}_{ji},\check{a}_{ji}], \tilde{b}_{ji} \in co\left[\hat{b}_{ji},\check{b}_{ji}\right]$, $\tilde{c}_{ij} \in co[\hat{c}_{ij},\check{c}_{ij}]$, and $\tilde{d}_{ij} \in co\left[\hat{d}_{ij},\check{d}_{ij}\right]$ such that

$$X-layer:\begin{cases} STM:\ \varepsilon\dot{e}_i^x(t) = -e_i^x(t) + \sum_{j=1}^{m}\tilde{a}_{ji}f_j\left(e_j^y(t)\right) \\ \qquad + \sum_{j=1}^{m}\tilde{b}_{ji}f_j\left(e_j^y(t-\tau(t))\right) + I_ih_i^s(t) + u_i(t), \\ LTM:\ \dot{h}_i^s(t) = -h_i^s(t) + g_i(e_i^x(t)),\ i = 1,2,\ldots,n, \end{cases} \tag{6c}$$

$$Y-layer:\begin{cases} STM:\ \varepsilon\dot{e}_j^y(t) = -e_j^y(t) + \sum_{i=1}^{n}\tilde{c}_{ij}g_i(e_i^x(t)) \\ \qquad + \sum_{i=1}^{n}\tilde{d}_{ij}g_i(e_i^x(t-\tau(t))) + J_jh_j^r(t) + \upsilon_j(t), \\ LTM:\ \dot{h}_j^r(t) = -h_j^r(t) + f_j\left(e_j^y(t)\right),\ j = 1,2,\ldots,m, \end{cases} \tag{6d}$$

where $f_j\left(e_j^y(t)\right) = f_j(\tilde{y}_j(t)) - f_j(y_j(t)), f_j\left(e_j^y(t-\tau(t))\right) = f_j(\tilde{y}_j(t-\tau(t))) - f_j(y_j(t-\tau(t)))$, similarly, $g_i(e_i^x(t)) = g_i(\tilde{x}_i(t)) - g_i(x_i(t))$, $g_i(e_i^x(t-\tau(t))) = g_i(\tilde{x}_i(t-\tau(t))) - g_i(x_i(t-\tau(t)))$.

Definition 3. Synchronization of system (1) and (2), that is to say, the trivial solution of system (6) or (7) is said to be globally asymptotically stable if for any given initial conditions they satisfy:

$$\lim_{x\to+\infty}\|e(t)\|^2 = 0,\ \lim_{x\to+\infty}\|h(t)\|^2 = 0$$

where $e(t) = ((e_x\ \ (t))^T, (e_y\ \ (t))^T)^T = \ (e_1^x(t), e_2^x(t),\ \ \ldots, e_n^x(t), e_1^y(t), \ldots, e_m\ y(t))^T$, $h(t) = \left((h_s(t))^T, (h_r(t))^T\right)^T = (e_1^s(t), e_2^s(t), \ldots, e_n^s(t), e_1^r(t), \ldots, e_m^r(t))^T$.

In our paper, let $U(t)$ and $V(t)$ be designed as follows:

Delay – dependent controller :

$$\left(U^T, V^T\right)^T = K_1e(t) + K_2e(t-\tau(t)), \tag{6e}$$

$$\text{Delay – independent controller}:\ \left(U^T, V^T\right)^T = K_1e(t), \tag{6f}$$

where K_i are the controller gains, $K_i = \begin{pmatrix} K_i^x & 0 \\ 0 & K_i^y \end{pmatrix}, K_i^x =$ diag$(k_{i1}^x$, $k_{i2}^x,\ \ldots,\ k_{in}^x), K_i^y = \text{diag}(k_{i1}^y,\ k_{i2}^y,\ \ldots,\ k_{im}^y), i = 1,\ 2, e(t-\tau(t)) = ((e_x(t-\tau(t)))^T,\ \ (e_y(t-\tau(t)))^T)^T = (e_1^x(t-\tau(t)),\ \ e_2^x(t-\tau(t)), \ldots, e_n^x(t-\tau(t)), e_1^y(t-\tau(t)), \ldots, e_m^y(t-\tau(t)))^T$.

3. Main results

Theorem 1. *Assume that (H1) and (H2) hold. Then two coupled delayed neural networks (2) and (4) or (3) and (5) can be synchronized with control input (8a), if there exist constants $r_1, r_2, r_3, r_4 > 0$, $(n+m)$-order positive definite diagonal matrix $Q > 0$ and $K_1,\ K_2$ such that $T > 0$, where*

$$T = \frac{I}{\varepsilon} - \frac{1}{\varepsilon}K_1 - \frac{r_1}{2\varepsilon}\left(\begin{pmatrix} 0 & \tilde{B} \\ \tilde{D} & 0 \end{pmatrix}\begin{pmatrix} L & 0 \\ 0 & \eta \end{pmatrix}\right)\left(\begin{pmatrix} 0 & \tilde{B} \\ \tilde{D} & 0 \end{pmatrix}\begin{pmatrix} L & 0 \\ 0 & \eta \end{pmatrix}\right)^T$$
$$-Q - \frac{1}{\varepsilon}\begin{pmatrix} 0 & \tilde{A} \\ \tilde{C} & 0 \end{pmatrix}\begin{pmatrix} L & 0 \\ 0 & \eta \end{pmatrix} - \frac{r_2}{2\varepsilon}\begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix}^T$$
$$-\frac{r_3}{2\varepsilon}\begin{pmatrix} L & 0 \\ 0 & \eta \end{pmatrix}^T\begin{pmatrix} L & 0 \\ 0 & \eta \end{pmatrix} - \frac{r_4}{2\varepsilon}K_2K_2^T.$$

Proof 1. For system (7), let the following Lyapunov–Krasovskii function $V(t, e(t))$ be

$$V(t, e(t)) = \frac{1}{2}\left(e^T(t)e(t) + h^T(t)h(t)\right) + \int_{t-\tau(t)}^{t}e^T(s)Qe(s)ds. \tag{6g}$$

In terms of assumption (H2), from (6) to (7), we have

$$\begin{aligned}
\dot{V}(t, e(t)) \leqslant\ & e_x^T(t)\tfrac{1}{\varepsilon}\Big[-e_x(t) + \tilde{A}f(e_y(t)) + \tilde{B}f(e_y(t-\tau(t))) + Ih_s(t) \\
& + K_1^xe_x(t) + K_2^xe_x(t-\tau(t))\Big] + h_s^T(t)[-h_s(t) + g(e_x(t))] \\
& + e_y^T(t)\tfrac{1}{\varepsilon}\Big[-e_y(t) + \tilde{C}g(e_x(t)) + \tilde{D}g(e_x(t-\tau(t))) + Jh_r(t) \\
& + K_1^ye_y(t) + K_2^ye_y(t-\tau(t))\Big] + h_r^T(t)[-h_r(t) + f(e_y(t))] \\
& + e^T(t)Qe(t) - (1-\gamma)e^T(t-\tau(t))Qe(t-\tau(t)) \\
=\ & -\tfrac{1}{\varepsilon}\left(e_x^T(t)e_x(t) + e_y^T(t)e_y(t)\right) + \tfrac{1}{\varepsilon}\left(e_x^T(t)\tilde{A}f(e_y(t)) + e_y^T(t)\tilde{C}g(e_x(t))\right) \\
& + \tfrac{1}{\varepsilon}\left(e_x^T(t)\tilde{B}f(e_y(t-\tau(t))) + e_y^T(t)\tilde{D}g(e_x(t-\tau(t)))\right) \\
& + \tfrac{1}{\varepsilon}\left(e_x^T(t)Ih_s(t) + e_y^T(t)Jh_r(t)\right) + \tfrac{1}{\varepsilon}\left(e_x^T(t)K_1^xe_x(t) + e_y^T(t)K_1^ye_y(t)\right) \\
& + \tfrac{1}{\varepsilon}\left(e_x^T(t)K_2^xe_x(t-\tau(t)) + e_y^T(t)K_2^ye_y(t-\tau(t))\right) \\
& - \left(h_s^T(t)h_s(t) + h_r^T(t)h_r(t)\right) + h_s^T(t)g(e_x(t)) + h_r^T(t)f(e_y(t)) \\
& + e^T(t)Qe(t) - (1-\gamma)e^T(t-\tau(t))Qe(t-\tau(t)) \\
=\ & -\tfrac{1}{\varepsilon}\left(e_x^T(t)e_x(t) + e_y^T(t)e_y(t)\right) + \tfrac{1}{\varepsilon}\left(e_x^T(t)\tilde{A}f(e_y(t)) + e_y^T(t)\tilde{C}g(e_x(t))\right) \\
& + \tfrac{1}{\varepsilon}\left(e_x^T(t)\tilde{B}f(e_y(t-\tau(t))) + e_y^T(t)\tilde{D}g(e_x(t-\tau(t)))\right) \\
& + \tfrac{1}{\varepsilon}\left(e_x^T(t)Ih_s(t) + e_y^T(t)Jh_r(t)\right) + \tfrac{1}{\varepsilon}\left(e_x^T(t)K_1^xe_x(t) + e_y^T(t)K_1^ye_y(t)\right) \\
& + \tfrac{1}{\varepsilon}\left(e_x^T(t)K_2^xe_x(t-\tau(t)) + e_y^T(t)K_2^ye_y(t-\tau(t))\right) \\
& - \left(h_s^T(t)h_s(t) + h_r^T(t)h_r(t)\right) + h_s^T(t)g(e_x(t)) + h_r^T(t)f(e_y(t)) \\
& + e^T(t)Qe(t) - (1-\gamma)e^T(t-\tau(t))Qe(t-\tau(t)) \\
=\ & -\tfrac{1}{\varepsilon}e^T(t)e(t) + \tfrac{1}{\varepsilon}e^T(t)\begin{pmatrix} 0 & \tilde{A} \\ \tilde{C} & 0 \end{pmatrix}\begin{pmatrix} g(e_x(t)) \\ f(e_y(t)) \end{pmatrix} \\
& + \tfrac{1}{\varepsilon}e^T(t)\begin{pmatrix} 0 & \tilde{B} \\ \tilde{D} & 0 \end{pmatrix}\begin{pmatrix} g(e_x(t-\tau(t))) \\ f(e_y(t-\tau(t))) \end{pmatrix} + \tfrac{1}{\varepsilon}e^T(t)\begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix}\begin{pmatrix} h_s(t) \\ h_r(t) \end{pmatrix} \\
& + \tfrac{1}{\varepsilon}e^T(t)K_1e(t) + \tfrac{1}{\varepsilon}e^T(t)K_2e(t-\tau(t)) + h^T(t)h(t) \\
& + h^T(t)\begin{pmatrix} g(e_x(t)) \\ f(e_y(t)) \end{pmatrix} + e^T(t)Qe(t) - (1-\gamma)e^T(t-\tau(t))Qe(t-\tau(t))
\end{aligned} \tag{10}$$

From Assumption (H1) and Lemma 1, there exist positive factors $r_1, r_2, r_3, r_4 > 0$ and n-order and m-order diagonal matrixs $L_{n\times n}$ and $\eta_{m\times m}$, respectively, satisfying

$$e^T(t)\begin{pmatrix}0 & \tilde{A}\\ \tilde{C} & 0\end{pmatrix}\begin{pmatrix}g(e_x(t))\\ f(e_y(t))\end{pmatrix} \leqslant e^T(t)\begin{pmatrix}0 & \tilde{A}\\ \tilde{C} & 0\end{pmatrix}\begin{pmatrix}L & 0\\ 0 & \eta\end{pmatrix}e(t) \tag{11}$$

$$\begin{aligned}&e^T(t)\begin{pmatrix}0 & \tilde{B}\\ \tilde{D} & 0\end{pmatrix}\begin{pmatrix}g(e_x(t-\tau(t)))\\ f(e_y(t-\tau(t)))\end{pmatrix} \leqslant \tfrac{1}{2r_1}e^T(t-\tau(t))e(t-\tau(t))\\ &+\tfrac{r_1}{2}e^T(t)\left(\begin{pmatrix}0 & \tilde{B}\\ \tilde{D} & 0\end{pmatrix}\begin{pmatrix}L & 0\\ 0 & \eta\end{pmatrix}\right)\left(\begin{pmatrix}0 & \tilde{B}\\ \tilde{D} & 0\end{pmatrix}\begin{pmatrix}L & 0\\ 0 & \eta\end{pmatrix}\right)^T e(t)\end{aligned} \tag{12}$$

$$\begin{aligned}&e^T(t)\begin{pmatrix}I & 0\\ 0 & J\end{pmatrix}\begin{pmatrix}h_s(t)\\ h_r(t)\end{pmatrix} \leqslant \tfrac{r_2}{2}e^T(t)\begin{pmatrix}I & 0\\ 0 & J\end{pmatrix}\begin{pmatrix}I & 0\\ 0 & J\end{pmatrix}^T\\ &e(t)+\tfrac{1}{2r_2}h^T(t)h(t)\end{aligned} \tag{13}$$

$$h^T(t)\begin{pmatrix}g(e_x(t))\\ f(e_y(t))\end{pmatrix} \leqslant \tfrac{1}{2r_3}h^T(t)h(t)+\tfrac{r_3}{2}e^T(t)\begin{pmatrix}L & 0\\ 0 & \eta\end{pmatrix}^T\begin{pmatrix}L & 0\\ 0 & \eta\end{pmatrix}e(t) \tag{14}$$

$$e^T(t)K_2e(t-\tau(t)) \leqslant \tfrac{r_4}{2}e^T(t)K_2K_2^Te(t)+\tfrac{1}{2r_4}e^T(t-\tau(t))e(t-\tau(t)). \tag{15}$$

Substituting (11)–(15) into (10) we have

$$\begin{aligned}\dot{V}(t,e(t)) \leqslant -e^T(t)\Bigg[&\tfrac{I}{\varepsilon}-\tfrac{1}{\varepsilon}\begin{pmatrix}K_1^x & 0\\ 0 & K_1^y\end{pmatrix}-\tfrac{r_1}{2\varepsilon}\left(\begin{pmatrix}0 & \tilde{B}\\ \tilde{D} & 0\end{pmatrix}\begin{pmatrix}L & 0\\ 0 & \eta\end{pmatrix}\right)\\ &\left(\begin{pmatrix}0 & \tilde{B}\\ \tilde{D} & 0\end{pmatrix}\begin{pmatrix}L & 0\\ 0 & \eta\end{pmatrix}\right)^T-Q-\tfrac{1}{\varepsilon}\begin{pmatrix}0 & \tilde{A}\\ \tilde{C} & 0\end{pmatrix}\begin{pmatrix}L & 0\\ 0 & \eta\end{pmatrix}\\ &-\tfrac{r_2}{2\varepsilon}\begin{pmatrix}I & 0\\ 0 & J\end{pmatrix}\begin{pmatrix}I & 0\\ 0 & J\end{pmatrix}^T-\tfrac{r_3}{2\varepsilon}\begin{pmatrix}L & 0\\ 0 & \eta\end{pmatrix}^T\begin{pmatrix}L & 0\\ 0 & \eta\end{pmatrix}-\tfrac{r_4}{2\varepsilon}K_2K_2^T\Bigg]\\ &e(t)+e(t-\tau(t))\left[\tfrac{I}{2\varepsilon r_1}+\tfrac{I}{2\varepsilon r_4}-(1-\gamma)Q\right]e(t-\tau(t))\\ &+h^T(t)\left[-I+\tfrac{I}{2\varepsilon r_2}+\tfrac{I}{2r_3}\right]h(t),\end{aligned} \tag{16}$$

where I is the identity matrix of appropriate dimension. It is easy to show that there are real numbers r_2 and r_3 such that

$$-I+\frac{I}{2\varepsilon r_2}+\frac{I}{2r_3}<0. \tag{17}$$

Letting

$$(1-\gamma)Q=\tfrac{I}{2\varepsilon r_1}+\tfrac{I}{2\varepsilon r_4}\ \lambda=min\left\{\lambda_{min}(T),\lambda_{min}\left(I-\tfrac{I}{2\varepsilon r_2}-\tfrac{I}{2r_3}\right)\right\} \tag{18}$$

From (16)–(18), it can be seen that

$$\dot{V}(t,e(t)) \leqslant -\lambda\left(\|e(t)\|^2+\|h(t)\|^2\right). \tag{19}$$

Moreover, in (19), the inequality holds if and only if $\|e(t)\|^2+\|h(t)\|^2=0$, i.e.,$\|e(t)\|^2=0$ and $\|h(t)\|^2=0$. Based on the Lyapunov stability theory, it can be concluded that

$$\lim_{x\to+\infty}\|e(t)\|^2=0,\ \lim_{x\to+\infty}\|h(t)\|^2=0$$

By Definition 3, the trivial solution of system (8) or (9) is globally asymptotically stable. That is to say, the neural networks (1) and (2) can be synchronized with the control input (8a). The proof is complete.

Remark 3. When the synapse strength of the external stimulus is constant for the STM state, system (1) is a continuous heteroassociative memory system [11,28,29]. Furthermore, when system (1) does not exhibit memristive, it is reduced to a general BAM neural network in [4,6].

Corollary 1. *Assume that* (*H*1) *and* (*H*2) *hold. When* $\tau(t)=\tau>0$, *then two coupled delayed neural networks (2) and (4) or (3) and (5) can be synchronized with control input (8a), if there exist constants* $r_1, r_2, r_3, r_4 > 0$, *positive definite diagonal matrix* $Q>0$ *and* K_1, K_2 *such that* $T>0$, *where*

$$\begin{aligned}T=&\tfrac{I}{\varepsilon}-\tfrac{1}{\varepsilon}K_1-\tfrac{r_1}{2\varepsilon}\left(\begin{pmatrix}0 & \tilde{B}\\ \tilde{D} & 0\end{pmatrix}\begin{pmatrix}L & 0\\ 0 & \eta\end{pmatrix}\right)\left(\begin{pmatrix}0 & \tilde{B}\\ \tilde{D} & 0\end{pmatrix}\begin{pmatrix}L & 0\\ 0 & \eta\end{pmatrix}\right)^T\\ &-Q-\tfrac{1}{\varepsilon}\begin{pmatrix}0 & \tilde{A}\\ \tilde{C} & 0\end{pmatrix}\begin{pmatrix}L & 0\\ 0 & \eta\end{pmatrix}-\tfrac{r_2}{2\varepsilon}\begin{pmatrix}I & 0\\ 0 & J\end{pmatrix}\begin{pmatrix}I & 0\\ 0 & J\end{pmatrix}^T\\ &-\tfrac{r_3}{2\varepsilon}\begin{pmatrix}L & 0\\ 0 & \eta\end{pmatrix}^T\begin{pmatrix}L & 0\\ 0 & \eta\end{pmatrix}-\tfrac{r_4}{2\varepsilon}K_2K_2^T.\end{aligned}$$

Proof 2. We obtain Corollary 1 directly from Theorem 1 by taking $Q=\frac{I}{2\varepsilon r_1}+\frac{I}{2\varepsilon r_4}$.

In the following, we will discuss synchronization problems of system (2) and (4) with another controller, which is delay-independent. Theoretically, the controller can be easily applied in engineering, because a controller with time delays may have a complex behaviour.

Theorem 2. *Assume that* (*H*1) *and* (*H*2) *hold. Then two coupled delayed neural networks (2) and (4) or (3) and (5) can be synchronized with control input (8b), if there exist constants* $r_1, r_2, r_3, r_4 > 0, (n+m)$*-order positive definite diagonal matrix* $Q>0$ *and* K_1, K_2 *such that* $T>0$, *where*

$$\begin{aligned}T=&\tfrac{I}{\varepsilon}-\tfrac{1}{\varepsilon}K_1-\tfrac{r_1}{2\varepsilon}\left(\begin{pmatrix}0 & \tilde{B}\\ \tilde{D} & 0\end{pmatrix}\begin{pmatrix}L & 0\\ 0 & \eta\end{pmatrix}\right)\left(\begin{pmatrix}0 & \tilde{B}\\ \tilde{D} & 0\end{pmatrix}\begin{pmatrix}L & 0\\ 0 & \eta\end{pmatrix}\right)^T\\ &-Q-\tfrac{1}{\varepsilon}\begin{pmatrix}0 & \tilde{A}\\ \tilde{C} & 0\end{pmatrix}\begin{pmatrix}L & 0\\ 0 & \eta\end{pmatrix}-\tfrac{r_2}{2\varepsilon}\begin{pmatrix}I & 0\\ 0 & J\end{pmatrix}\begin{pmatrix}I & 0\\ 0 & J\end{pmatrix}^T\\ &-\tfrac{r_3}{2\varepsilon}\begin{pmatrix}L & 0\\ 0 & \eta\end{pmatrix}^T\begin{pmatrix}L & 0\\ 0 & \eta\end{pmatrix}.\end{aligned}$$

Proof 3. Taking $(1-\gamma)Q=\frac{I}{2\varepsilon r_1}$, we can easily obtain above result.

Furthermore, when $\tau(t)=\tau>0$, the controller is designed as (8b), we can obtain the corollary below:

Corollary 2. *Assume that* (*H*1) *and* (*H*2) *hold. Then* $\tau(t)=\tau>0$, *then two coupled delayed neural networks (2) and (4) or (3) and (5) can be synchronized with control inputs (8b), if there exist constants* $r_1, r_2, r_3, r_4 > 0$, *positive definite diagonal matrix* $Q>0$ *and* K_1, K_2 *such that* $T>0$, *where* $T=\frac{I}{\varepsilon}-\frac{1}{\varepsilon}K_1-\frac{r_1}{2\varepsilon}\left(\begin{pmatrix}0 & \tilde{B}\\ \tilde{D} & 0\end{pmatrix}\begin{pmatrix}L & 0\\ 0 & \eta\end{pmatrix}\right)$ $\left(\begin{pmatrix}0 & \tilde{B}\\ \tilde{D} & 0\end{pmatrix}\begin{pmatrix}L & 0\\ 0 & \eta\end{pmatrix}\right)^T-Q-\frac{1}{\varepsilon}\begin{pmatrix}0 & \tilde{A}\\ \tilde{C} & 0\end{pmatrix}\ \begin{pmatrix}L & 0\\ 0 & \eta\end{pmatrix}-\frac{r_2}{2\varepsilon}\begin{pmatrix}I & 0\\ 0 & J\end{pmatrix}\begin{pmatrix}I & 0\\ 0 & J\end{pmatrix}^T-$ $\frac{r_3}{2\varepsilon}\begin{pmatrix}L & 0\\ 0 & \eta\end{pmatrix}^T\begin{pmatrix}L & 0\\ 0 & \eta\end{pmatrix}$.

Remark 4. Thus, our results extend known results and can be applied to much wider situations.

4. A numerical example

In the following, we give some numerical simulations to illustrate the results above. Consider the following memristor-based competitive BAM neural networks with different time scales:

$$X-layer: \begin{cases} STM: \ \varepsilon\dot{x}_i(t) = -x_i(t) + \sum_{j=1}^{m} a_{ji}(x_i(t)) f_j(y_j(t)) \\ \qquad + \sum_{j=1}^{m} b_{ji}(x_i(t)) f_j(y_j(t-\tau(t))) + I_i s_i(t), \\ LTM: \ \dot{s}_i(t) = -s_i(t) + g_i(x_i(t)), \ i = 1,2, \end{cases} \tag{20a}$$

$$Y-layer: \begin{cases} STM: \ \varepsilon\dot{y}_j(t) = -y_j(t) + \sum_{i=1}^{n} c_{ij}(y_j(t)) g_i(x_i(t)) \\ \qquad + \sum_{i=1}^{n} d_{ij}(y_j(t)) g_i(x_i(t-\tau(t))) + J_j r_j(t), \\ LTM: \ \dot{r}_j(t) = -r_j(t) + f_j(y_j(t)), \ j = 1, \end{cases} \tag{20b}$$

where $\varepsilon = 0.8, \tau(t) = 0.5|sint|, f(x(t)) = tanh(x(t)), g(y(t)) = tanh(y(t)), I_1 = 0.1, I_2 = 0.3, J_1 = 0.3$, with initial values $x_1(\theta) = -0.4$, $x_2(\theta) = 0.5, s_1(\theta) = 0.5, s_2(\theta) = 0.5, y_1(\theta) = 0.5, r_1(\theta) = 0.5, \forall\theta \in [-0.5, 0]$.

$$a_{11}(x_1(t)) = \begin{cases} 2.5, & |x_1| > 1, \\ -1.0, & |x_1| \leqslant 1, \end{cases} \quad b_{11}(x_1(t)) = \begin{cases} -2.0, & |x_1| > 1, \\ -1.0, & |x_1| \leqslant 1, \end{cases}$$

$$a_{12}(x_2(t)) = \begin{cases} -0.15, & |x_2| > 1, \\ -0.1, & |x_2| \leqslant 1, \end{cases} \quad b_{12}(x_2(t)) = \begin{cases} -0.3, & |x_2| > 1, \\ -0.2, & |x_2| \leqslant 1, \end{cases}$$

$$c_{11}(y_1(t)) = \begin{cases} 2.5, & |y_1| > 1, \\ -0.1, & |y_1| \leqslant 1, \end{cases} \quad c_{21}(y_1(t)) = \begin{cases} -0.15, & |y_1| > 1, \\ -0.1, & |y_1| \leqslant 1, \end{cases}$$

$$d_{11}(y_1(t)) = \begin{cases} -0.3, & |y_1| > 1, \\ -0.2, & |y_1| \leqslant 1, \end{cases} \quad d_{21}(y_1(t)) = \begin{cases} -2.0, & |y_1| > 1, \\ -1.5, & |y_1| \leqslant 1, \end{cases}$$

The corresponding response system is as follows:

$$X-layer: \begin{cases} STM: \ \varepsilon\dot{\tilde{x}}_i(t) = -\tilde{x}_i(t) + \sum_{j=1}^{m} \tilde{a}_{ji} f_j(\tilde{y}_j(t)) \\ \qquad + \sum_{j=1}^{m} \tilde{b}_{ji} f_j(\tilde{y}_j(t-\tau(t))) + I_i \tilde{s}_i(t) + u_i(t), \\ LTM: \ \dot{\tilde{s}}_i(t) = -\tilde{s}_i(t) + g_i(\tilde{x}_i(t)), \ i = 1,2, \end{cases} \tag{21a}$$

$$Y-layer: \begin{cases} STM: \ \varepsilon\dot{\tilde{y}}_j(t) = -\tilde{y}_j(t) + \sum_{i=1}^{n} \tilde{c}_{ij} g_i(\tilde{x}_i(t)) \\ \qquad + \sum_{i=1}^{n} \tilde{d}_{ij} g_i(\tilde{x}_i(t-\tau(t))) + J_j \tilde{r}_j(t) + v_j(t), \\ LTM: \ \dot{\tilde{r}}_j(t) = -\tilde{r}_j(t) + f_j(\tilde{y}_j(t)), \ j = 1, \end{cases} \tag{21b}$$

with the initial values $\tilde{x}_1(\theta) = 0.3$, $\tilde{x}_2(\theta) = -0.5$, $\tilde{s}_1(\theta) = 0.5$, $\tilde{s}_2(\theta) = 0.5$, $\tilde{y}_1(\theta) = 0.5$, $\tilde{r}_1(\theta) = 0.5$. $\forall\theta \in [-0.5, 0]$, $\left(U^T, V^T\right)^T = \begin{pmatrix} K_1^x & 0 \\ 0 & K_1^y \end{pmatrix} e(t) + \begin{pmatrix} K_2^x & 0 \\ 0 & K_2^y \end{pmatrix} e(t-\tau(t))$,

$$K_1^x = \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix}, \ K_2^x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ K_1^y = -4, \ K_2^y = 1.$$

Applying controller (8a), we can compute $\begin{pmatrix} 0 & \tilde{A} \\ \tilde{C} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & -0.1 \\ -1 & 0 & 0 \\ -0.1 & 0 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1.28125 & 1.25 & 0.125 \\ 1.25 & 1.85625 & -0.1875 \\ 0.125 & -0.1875 & 0.48 \end{pmatrix} > 0$; furthermore, $\lambda_{1,\,2,\,3}(T) = 0.1471,\ 0.617,\ 2.8534$, respectively. Set $r_1 = 1, r_2 = 1, r_3 = \frac{3}{2}, r_4 = 2, \gamma = \frac{1}{2}$, according to Theorem 1, the response system (20) and the drive system (21) with the controllers $u(t)$ and $v(t)$ can be globally asymptotically synchronized. At the same time, Fig. 1(A), Fig. 2(A), and Fig. 3A) show time evolutions of synchronization errors of fast neural state variables between drive systems (20) and response systems (21), respectively; Fig. 1B), Fig. 2B), and Fig. 3B) described synchronization errors of the slow activity of unsupervised synaptic modifications by input, respectively. When controller (8b) is applied in the response system (20) and the drive system (21), similar numerical results show the feasibility of our theoretical results.

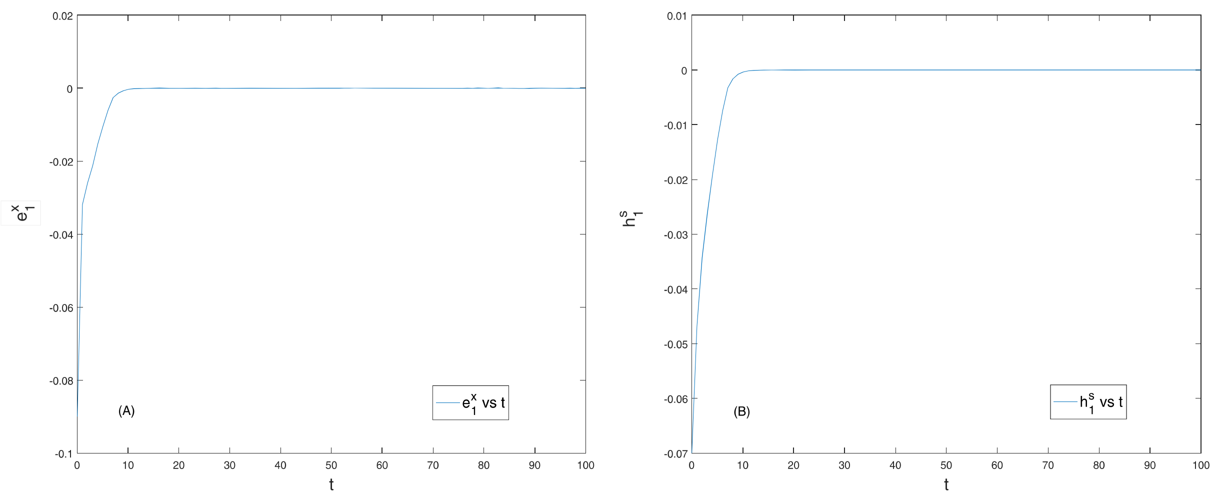

Fig. 1. Time evolutions of synchronization errors $e_1^x(t), h_1^s(t)$ for system (20) and (21).

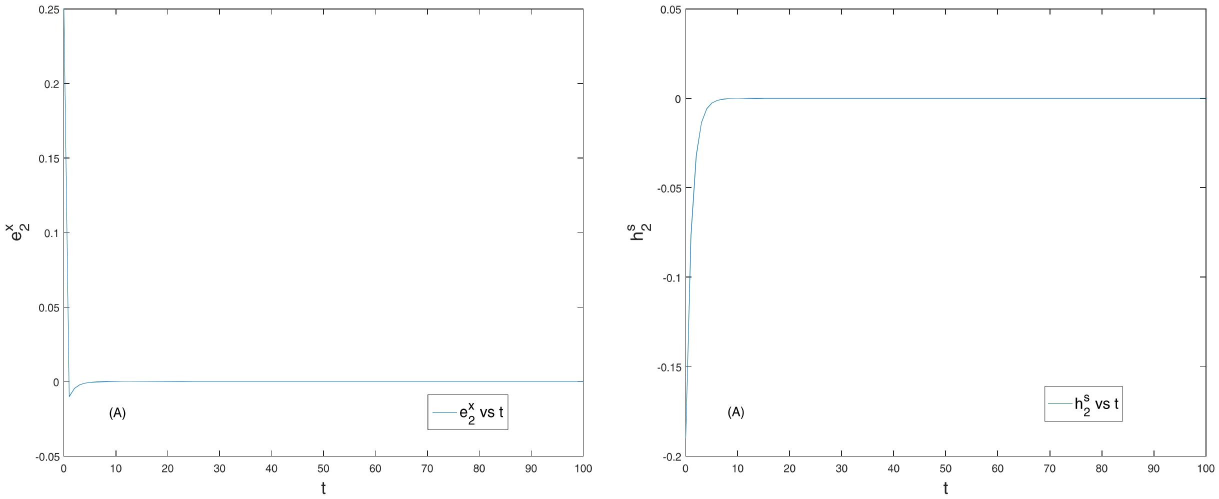

Fig. 2. Time evolutions of synchronization errors $e_2^x(t), h_2^s(t)$ for system (20) and (21).

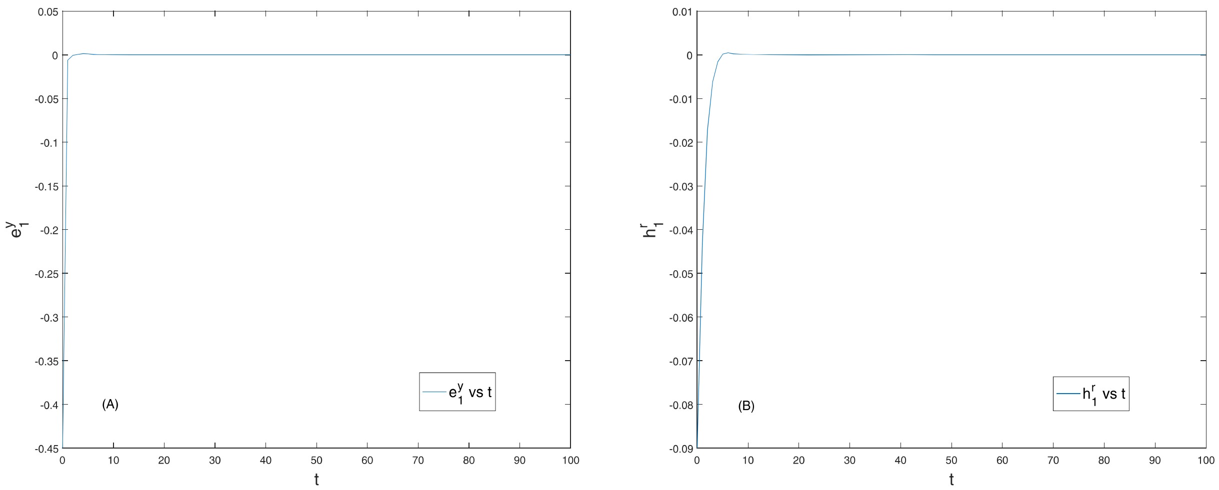

Fig. 3. Time evolutions of synchronization error $e_1^y(t), h_1^r(t)$ for system (20) and (21).

5. Conclusions

In this paper, we propose a double-layer memristor-based competitive BAM neural network with different time scales. Based on differential inclusions and Lyapunov functional, we explore the synchronization problem of this kind of neural network. Firstly, a controller with time delays is designed to achieve the synchronization of memristor-based competitive BAM neural networks and novel conditions to ensure synchronization of the drive system and the corresponding response system are given. Secondly, a simpler and easier applicable controller without time delay is discussed to reach synchronization, which is delay-independent. This controller can be easily applied in engineering, because a controller with time delays have a more complex behaviour. Finally, a numerical example demonstrates the effectiveness and feasibility of our results. When the synapse strength of the external stimulus is constant, system (1) is a continuous hetero-associative memory system [11,32,33]. Furthermore, when the synapse strength of system (1) is fixed, it is reduced to a classic BAM neural network, such as [4,6]. Uncertanity and random noise exist widely in nature, in social systems and in biological systems. In future, we will furthermore discuss memristor-based competitive BAM neural networks with different time scales under impulsice effects or stochastic disturbances.

CRediT authorship contribution statement

Yong Zhao: Conceptualization, Methodology, Software, Writing - original draft. **Shanshan Ren:** Formal analysis, Data curation, Visualization. **Jürgen Kurths:** Writing - review & editing.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

[1] A.L. Hodgkin, A.F. Huxley, A quantitative description of membrane current and its application to conduction and excitation in nerve, J. Physiol. 117 (1952) 500–544.
[2] W.S. McCulloch, W.H. Pitts, A logical calculus of the ideas immanent in nervous activity, Bull. Math. Biophys. 5 (1943) 115–133.
[3] J.J. Hopfield, Neural networks and physical systems with emergent collective computational abilities, Proc. Nat. Acad. Sci. USA 79 (1982) (1982) 2554–2558.
[4] B. Kosko, Bidirectional associative memories, IEEE Trans. Syst. Man Cybern. 18 (1988) 49–60.
[5] M.A. Cohen, S. Grossberg, Absolute stability and global pattern formation and parallel memory storage by competitive neural networks, IEEE Trans. Syst. Man Cybern. 13 (1983) 815–821.
[6] S.J. Guo, L.H. Huang, Periodic oscillation for a class of BAM neural networks with variable coefficients, Nonlinear Anal. Real World Appl. 6 (2005) 545–561.
[7] Z. Wang, Y. Liu, X. Liu, On global asymptotic stability of neural networks with discrete and distributed delays, Phys. Lett. A 345 (2005) 299–308.
[8] C.J. Xu, X.H. Tang, P.L. Li, existence and global stability of almost automorphic solutions for shunting inhibitory cellular neural networks with time-varying delays in leakage terms on time scales, J. Appl. Anal. Comput. 8 (2018) 1033–1049.
[9] K. Zhong, Q.Q. Yang, S. Zhu, New algebraic conditions for ISS of memristive neural networks with variable delays, Neural Comput. Appl. 28 (2017) 2089–2097.
[10] Y. Zhao, J. Kurths, L.X. Duan, Input-to-State stability analysis for memristive Cohen-Grossberg-type neural networks with variable time delays, Chaos Solitons Fract. 114 (2018) 364–369.
[11] Y. Zhao, J. Kurths, L.X. Duan, Input-to-State stability analysis for memristive BAM neural networks with variable time delays, Phys. Lett. A 383 (2019) 1143–1150.
[12] D.O. Hebb, The Organization of Behavior, Wiley, New York, 1949.
[13] M. Lemmon B. Kumar, Emulating the dynamics for a class of laterally inhibited neural networks, Neural Networks 2 (1989) 193–214.
[14] A. Meyer-Bäse, F. Ohl, H. Scheich, Singular perturbation analysis of competitive neural networks with different time scales, Neural Comput. 8 (1996) 1731–1742.
[15] A. Meyer-Bäse, S.S. Pilyugin, A. Wismler, S. Foo, Local exponential stability of competitive neural networks with different time scales, Eng. Appl. Artif. Intell. 17 (2004) 227–232.
[16] Q.T. Gan, R. Xu, X.B. Kang, Synchronization of unknown chaotic delayed competitive neural networks with different time scales based on adaptive control and parameter identification, Nonlinear Dyn. 67 (2012) 1893–1902.
[17] X.Y. Lou, B.T. Cui, Synchronization of competitive neural networks with different time scales, Phys. A 380 (2007) 563–576.
[18] Y.C. Shi, P.Y. Zhu, Synchronization of memristive competitive neural networks with different time scales, Neural Comput. Appl. 25 (2014) 1163–1168.
[19] A. Pikovsky, M. Rosenblum, J. Kurths, Synchronization: A Universal Concept in Nonlinear Sciences, Cambridge University Press, Cambridge, 2011.
[20] G. Buzsaki, Rhythms of the Brain, Oxford University Press, Oxford, 2011.
[21] J.Q. Lu, X. Guo, T.W. Huang, Z. Wang, Consensus of signed networked multi-agent systems with nonlinear coupling and communication delays, Appl. Math. Comput. 350 (2019) 153–162.
[22] Y.Y. Li, J.G. Lou, Z. Wang, F. Alsaadi, Synchronization of dynamical networks with nonlinearly coupling function under hybrid pinning impulsive controllers, J. Franklin Inst. 355 (2018) 6520–6530.
[23] H.W. Chen, J.L. Liang, Local Synchronization of interconnected Boolean networks with stochastic disturbances, IEEE Trans. Neural Networks Learn. Syst. 31 (2020) 452–463.
[24] P. Fries, J.H. Reynolds, J.H. Rorie, R. Desimore, Modulation of oscillatory neural synchronizaiton by selective visula attention, Science 291 (2001) 1560–1563.
[25] A. Schnitzler, J. Gross, Normal and pathological oscillatory communication in the brain, Nat. Rev. Neurosci. 6 (2005) 285–296.
[26] P.J. Uhlhaas, W. Singer, Neural synchrony in brain review disorders: relevance for cognitive dysfunctions and pathophysiology, Neuron 52 (2006) 155–168.
[27] B. Pesaran, J. Pezaris, M. Sahani, P. Mitra, R. Andersen, Temporal structure in neuronal activity during working memory in macaque parietal cortex, Nature 5 (2002) 805–811.
[28] D.B. Strukov, G.S. Snider, G.R. Stewart, R.S. Williams, The missing memristor found, Nature 453 (2008) 80–83.
[29] R. Legenstein, Computer science: nanoscale connections for brain-like circuits, Nature 521 (2015) 37–38.
[30] J.D. Cao, R.X. Li, fixed-time synchronization of delayed memristor-based recurrent neural networks, Sci. China Inf. Sci. 60 (2017) 032201.
[31] R.Y. Wei, J.D. Cao, A. Alsaedi, Fixed-time synchronization of memristive Cohen-Grossberg neural networks with impulsive effects, Int. J. Control Autom. Syst. 16 (2018) 2214–2224.
[32] C. Chen, L.X. Li, H.P. Peng, Y.X. Yang, Fixed-time synchronization of memristor-based BAM neural networks with time-varying discrete delay, Neural Networks 96 (2017) 47–54.
[33] C. Chen, L.X. Li, H.P. Peng, Y.X. Yang, Adaptive synchronization of memristor-based BAM neural networks with mixed delays, Appl. Math. Comput. 322 (2018) 100–110.
[34] A.F. Filippov, Differential Equations with Discontinuous Right-hand Sides, Kluwer, Dordrecht, 1988.
[35] J.P. Aubin, A. Cellina, Differential Inclusions, Springer, Berlin, 1984.
[36] F.H. Clarke, Y.S. Ledyaev, R.J. Stem, R.R. Wolenski, Nonsmooth Analysis and Control Theory, Springer, New York, 1998.

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